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Regular products of the single-hook characters of the unitary group

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Abstract. Regular products of single-hook characters of the unitary group are defined. Their expansion in terms of irreducible characters is discussed and illustrated, along with the inverse of this expansion for all characters labelled by partitions of n with $n = 5$ and $n = 9$. It is shown that these regular products form an integral basis for the vector space of homogeneous functions.

1. Introduction

The character of each irreducible representation of the unitary group $U(N)$ may be conveniently denoted by $\{\boldsymbol{\lambda}\}$ where $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots)$ is a partition of n for some n. It is well known that such a character possesses determinantal expansions as a sum of products of symmetric characters $\{m\}$ (Littlewood 1940, p 88) and as a sum of products of antisymmetric characters $\{1^m\}$ (Littlewood 1940, p 89). Furthermore, there exists a reduced determinantal expansion of $\{\boldsymbol{\lambda}\}$ as a sum of products of single-hook characters $\{1+a, 1^b\}$ (Littlewood 1940, p 94, Macdonald 1979, p 68).

The first two expansions of the characters $\{\boldsymbol{\lambda}\}$, for all partitions $\boldsymbol{\lambda}$ of n, possess inverses whereby each product of symmetric characters and each product of antisymmetric characters of total weight n may be expressed as a sum of these irreducible characters $\{\boldsymbol{\lambda}\}$. In contrast to this, the third expansion does not possess an inverse, even though each product of single-hook characters of total weight n may be expressed as a sum of irreducible characters $\{\boldsymbol{\lambda}\}$ by means of the Littlewood–Richardson rule (Littlewood 1940, p 94).

The origin of the deficiency lies in the fact that the reduced determinantal expansion of $\{\boldsymbol{\lambda}\}$ gives rise to more products of single-hook characters of total weight n than there are partitions of n. This may be remedied by restricting these products to those which are regular in the sense that the individual hooks in the product constitute a regular Young diagram corresponding to some partition $\boldsymbol{\lambda}$ of n. This is explained in the next section where the notation and terminology is established.

The main concern of this paper is then to demonstrate, as is done in § 3, that these regular products of single-hook characters form a complete linearly independent set, and to give in tabular form both their expansion in terms of irreducible characters $\{\boldsymbol{\lambda}\}$, and the inverse expansion of $\{\boldsymbol{\lambda}\}$ in terms of these regular products.

‡ Professor Jahn died in October, 1979, at which time he was collaborating with one of us (NGEl-S) on the work reported here. Calculations found amongst his papers form the basis of this publication.

In § 4 the connection between these results and the classical results concerning symmetric functions is discussed. In particular, it is shown that the definition of regular products introduced in § 2 serves to provide a new integral basis (Hall 1959, Stanley 1971) for the vector space of homogeneous symmetric functions. Furthermore, an additional dual basis is also obtained.

A preliminary note (El-Sharkaway *et al* 1980) on this work has appeared elsewhere.

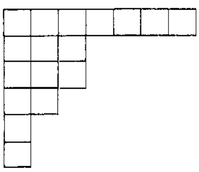
2. Notation

The partition of n into p non-vanishing parts $\lambda_1, \lambda_2, \ldots, \lambda_p$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p > 0$ is denoted by $\boldsymbol{\lambda} = (\lambda_1, \lambda_2, \ldots, \lambda_p)$. In such a case the weight of $\boldsymbol{\lambda}$ is said to be $n = \lambda_1 + \lambda_2 + \ldots + \lambda_p$, and it is then sometimes convenient to write $\boldsymbol{\lambda} \vdash n$. Each such partition specifies a regular Young diagram consisting of n boxes arranged in left-adjusted rows such that the ith row contains λ_i boxes. The fact that the same diagram consists of n boxes arranged in top-adjusted columns such that the jth column contains $\tilde{\lambda}_k$ boxes serves to define the conjugate partition $\tilde{\boldsymbol{\lambda}} = (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_q)$ where q is the number of columns. Furthermore, the same diagram also consists of n boxes arranged in diagonally adjusted right-angled hooks such that the kth hook contains $(\lambda_k + \tilde{\lambda}_k - 2k + 1)$ boxes arranged in such a way that the hook consists of a single box on the main diagonal together with an arm of length $a_k = \lambda_k - k$ and a leg of length $b_k = \tilde{\lambda}_k - k$. The length of the kth hook is thus $h_k = a_k + b_k + 1$, and the number of such hooks is the Frobenius rank, r, of the corresponding partition (Littlewood 1940, p 60). The partition $\boldsymbol{\lambda}$ thus defines and is defined by the Frobenius symbol

$$\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} a_1 a_2 \ldots a_r \\ b_1 b_2 \ldots b_r \end{pmatrix}. \tag{2.1}$$

The regularity of the Young diagram is ensured by the conditions $a_1 > a_2 > \ldots > a_r \geqslant 0$ and $b_1 > b_2 > \ldots > b_r \geqslant 0$.

This notation is exemplified in the case of the regular Young diagram



corresponding to the partition $\boldsymbol{\lambda} = (73^2 21^2)$, for which the conjugate partition is $\tilde{\boldsymbol{\lambda}} = (6431^4)$, whilst the Frobenius symbol is $\begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix} = \begin{pmatrix} 6 & 1 & 0 \\ 5 & 2 & 0 \end{pmatrix}$. The constituent hooks are of lengths given by $\boldsymbol{h} = (12, 4, 1)$ and the Frobenius rank is 3.

Each partition $\boldsymbol{\lambda}$ serves to label the character of an irreducible representation of the unitary group U(N) which is the S-function $\{\boldsymbol{\lambda}\}$ of the eigenvalues of the appropriate group element (Littlewood 1940, p 222, King and Plunkett 1976). Following Hall (1959) and Stanley (1971), such an S-function is also denoted by

$$e_{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}\}. \tag{2.2}$$

Each such partition also serves to label characters $S(\boldsymbol{\lambda})$ and $A(\boldsymbol{\lambda})$ (Braunschweig and Hecht 1978) of representations which are, in general, reducible. These characters are just the multiplicatively defined classical symmetric functions $h_{\boldsymbol{\lambda}}$ and $a_{\tilde{\boldsymbol{\lambda}}}$ (Littlewood 1940, p 104, Hall 1959, Stanley 1971, Macdonald 1979, pp 12, 15),

$$h_{\boldsymbol{\lambda}} = S(\boldsymbol{\lambda}) = \{\lambda_1\}\{\lambda_2\}\ldots\{\lambda_p\}, \tag{2.3}$$

$$a_{\tilde{\boldsymbol{\lambda}}} = A(\boldsymbol{\lambda}) = \{1^{\tilde{\lambda}_1}\}\{1^{\tilde{\lambda}_2}\}\ldots\{1^{\tilde{\lambda}_q}\}. \tag{2.4}$$

Clearly these products, involving as they do symmetric characters $\{\lambda_i\}$ and antisymmetric characters $\{1^{\tilde{\lambda}_i}\}$, exploit the row and column structure, respectively, of the Young diagram specified by the partition $\boldsymbol{\lambda}$. In precisely the same way it is possible to define new characters $R(\boldsymbol{\lambda})$, and correspondingly new symmetric functions $r_{\boldsymbol{\lambda}}$, by

$$r_{\boldsymbol{\lambda}} = R(\boldsymbol{\lambda}) = \{\lambda_1, 1^{\tilde{\lambda}_1-1}\}\{\lambda_2-1, 1^{\tilde{\lambda}_2-2}\}\ldots\{\lambda_r-r+1, 1^{\tilde{\lambda}_r-r}\},$$

which exploit the hook structure of the Young diagram specified by $\boldsymbol{\lambda}$. Such a product involves the single-hook characters

$$\{\lambda_k-k+1, 1^{\tilde{\lambda}_k-k}\} = \begin{Bmatrix} a_k \\ b_k \end{Bmatrix}, \tag{2.5}$$

so that in the notation of (2.1)

$$r_{\boldsymbol{\lambda}} = R(\boldsymbol{\lambda}) = \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix}\begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix}\cdots\begin{Bmatrix} a_r \\ b_r \end{Bmatrix}. \tag{2.6}$$

Such a product of single-hook characters is said to be regular, since the regular Young diagram corresponding to $\boldsymbol{\lambda}$ is recovered by diagonally adjusting the right-angled hooks $\begin{pmatrix} a_1 \\ b_1 \end{pmatrix}, \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}, \ldots, \begin{pmatrix} a_r \\ b_r \end{pmatrix}$ appearing in the product. This confirms the one-to-one correspondence between the symmetric functions $r_{\boldsymbol{\lambda}}$ and the partitions $\boldsymbol{\lambda}$.

In what follows, it is necessary to make use of an ordering scheme for the partitions $\boldsymbol{\lambda}$ of n which is appropriate to the use of the Frobenius symbol notation (2.1). If $\boldsymbol{\lambda} = \begin{pmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{pmatrix}$ and $\boldsymbol{h} = \boldsymbol{a} + \boldsymbol{b} + \mathbf{1}$ where $\boldsymbol{\lambda}$ has Frobenius rank r and $\mathbf{1} = (1^r)$, whilst $\boldsymbol{\lambda}' = \begin{pmatrix} \boldsymbol{a}' \\ \boldsymbol{b}' \end{pmatrix}$ and $\boldsymbol{h}' = \boldsymbol{a}' + \boldsymbol{b}' + \mathbf{1}'$ where $\boldsymbol{\lambda}'$ has Frobenius rank r' and $\mathbf{1}' = (1^{r'})$, and if $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}'$ are both partitions of n, then $\boldsymbol{\lambda}$ is said to precede $\boldsymbol{\lambda}'$ if the first non-vanishing difference $r'-r$, $h_k - h_k'$ for $k = 1, 2, \ldots, r$, $a_k - a_k'$ for $k = 1, 2, \ldots, r$ is positive. Thus the partitions are ordered by rank, and then lexicographically by hook lengths and finally lexicographically by arm lengths.

In the case of $n = 9$, for example, the ordered list of partitions is as follows:

$9, 81, 71^2, 61^3, 51^4, 41^5, 31^6, 21^7, 1^9,$

$72, 621, 521^2, 421^3, 321^4, 2^21^5,$

$63, 531, 52^2, 431^2, 42^21, 3^21^3, 32^21^2, 2^31^3,$

$54, 4^21, 432, 3^221, 32^3, 2^41$ and

3^3, corresponding to the Frobenius symbols

$$\begin{pmatrix} 8 \\ 0 \end{pmatrix}, \begin{pmatrix} 7 \\ 1 \end{pmatrix}, \begin{pmatrix} 6 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 3 \end{pmatrix}, \begin{pmatrix} 4 \\ 4 \end{pmatrix}, \begin{pmatrix} 3 \\ 5 \end{pmatrix}, \begin{pmatrix} 2 \\ 6 \end{pmatrix}, \begin{pmatrix} 1 \\ 7 \end{pmatrix}, \begin{pmatrix} 0 \\ 8 \end{pmatrix}$$

$$\binom{60}{10}, \binom{50}{20}, \binom{40}{30}, \binom{30}{40}, \binom{20}{50}, \binom{10}{60}$$

$$\binom{51}{10}, \binom{41}{20}, \binom{40}{21}, \binom{31}{30}, \binom{30}{31}, \binom{21}{40}, \binom{20}{41}, \binom{10}{51}$$

$$\binom{42}{10}, \binom{32}{20}, \binom{31}{21}, \binom{21}{31}, \binom{20}{32}, \binom{10}{42} \text{ and}$$

$$\binom{210}{210}.$$

3. The expansion of regular products of single-hook characters

It is well known that the characters $\{\boldsymbol{\lambda}\}$ of U(N) possess the determinantal expansions (Littlewood 1940, pp 88, 89)

$$e_{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}\} = |\{\lambda_i - i + j\}| = |h_{\lambda_i - i + j}|, \tag{3.1}$$

$$e_{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}\} = |\{1^{\tilde{\lambda}_j + i - j}\}| = |a_{\tilde{\lambda}_j + i - j}|, \tag{3.2}$$

where the ijth element of each determinant has been displayed with i and j ranging over the values $1, 2, \ldots, p$ and $1, 2, \ldots, q$ in (3.1) and (3.2) respectively. The expansion of these determinants and the commutativity of products of characters leads to the formulae

$$e_{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}\} = \sum_{\boldsymbol{\mu} \vdash n} B_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} S(\boldsymbol{\mu}) = \sum_{\boldsymbol{\mu} \vdash n} B_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} h_{\boldsymbol{\mu}}, \tag{3.3}$$

$$e_{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}\} = \sum_{\boldsymbol{\mu} \vdash n} B_{\tilde{\boldsymbol{\lambda}}}^{\tilde{\boldsymbol{\mu}}} A(\boldsymbol{\mu}) = \sum_{\boldsymbol{\mu} \vdash n} B_{\tilde{\boldsymbol{\lambda}}}^{\tilde{\boldsymbol{\mu}}} a_{\tilde{\boldsymbol{\mu}}}, \tag{3.4}$$

where the coefficients $B_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}}$ are integers which have been tabulated, for example, by Blaha (1969).

It is easy to see that if the partitions are ordered lexicographically with respect to the labels $\boldsymbol{\lambda}$ and $\tilde{\boldsymbol{\lambda}}$, the matrices of coefficients in (3.3) and (3.4) are lower triangular with each diagonal element equal to 1. It follows that the inverse matrices exist. They may be found directly by using the appropriate special cases of the Littlewood–Richardson rule (Littlewood 1940, p 94) for the multiplication of S-functions to evaluate the products

$$h_{\boldsymbol{\lambda}} = S(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\mu} \vdash n} K_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} \{\boldsymbol{\mu}\} = \sum_{\boldsymbol{\mu} \vdash n} K_{\boldsymbol{\lambda}}^{\boldsymbol{\mu}} e_{\boldsymbol{\lambda}}, \tag{3.5}$$

$$a_{\tilde{\boldsymbol{\lambda}}} = A(\boldsymbol{\lambda}) = \sum_{\boldsymbol{\mu} \vdash n} K_{\tilde{\boldsymbol{\lambda}}}^{\tilde{\boldsymbol{\mu}}} \{\boldsymbol{\mu}\} = \sum_{\boldsymbol{\mu} \vdash n} K_{\tilde{\boldsymbol{\lambda}}}^{\tilde{\boldsymbol{\mu}}} e_{\boldsymbol{\mu}}. \tag{3.6}$$

The coefficients appearing in (3.5) are the elements of Kostka's matrix K which have been tabulated not only by Kostka (1882, 1907), but also by, for example, Blaha (1969).

To generalise this work it is tempting to consider the reduced determinantal expansion (Littlewood 1940, p 112)

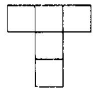
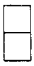
$$e_{\boldsymbol{\lambda}} = \{\boldsymbol{\lambda}\} = |\{\lambda_i - i + 1, 1^{\tilde{\lambda}_j - j}\}| = \left| \begin{Bmatrix} a_i \\ b_j \end{Bmatrix} \right|, \tag{3.7}$$

where now i and j range over the values $1, 2, \ldots, r$. Unfortunately, whilst this formula is the natural generalisation of (3.1) and (3.2) involving single-hook characters, it does not lead to an expansion of the form (3.3) and (3.4), since the products of single-hook characters which emerge include some which are not regular, in the sense that they do not correspond to $R(\boldsymbol{\mu})$ for any partition $\boldsymbol{\mu}$.

For example,

$$e_{3,2}=\{3,2\}=\begin{Bmatrix}2 & 0\\ 1 & 0\end{Bmatrix}=\begin{vmatrix}\begin{Bmatrix}2\\1\end{Bmatrix} & \begin{Bmatrix}2\\0\end{Bmatrix}\\ \begin{Bmatrix}0\\1\end{Bmatrix} & \begin{Bmatrix}0\\0\end{Bmatrix}\end{vmatrix}$$

$$=\begin{Bmatrix}2\\1\end{Bmatrix}\begin{Bmatrix}0\\0\end{Bmatrix}-\begin{Bmatrix}2\\0\end{Bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix}$$

where the first term is just $R(3,2)$, since it corresponds to the regular Young diagram , but the second term corresponds to an irregular diagram  when the constituent hooks  and  are diagonally adjusted.

It is therefore necessary to adopt a different approach avoiding irregular diagrams. This may be accomplished by evaluating the regular products of single-hook characters, (2.6), in the form

$$r_{\lambda}=R(\boldsymbol{\lambda})=\sum_{\mu\vdash n}H^{\mu}_{\lambda}\{\boldsymbol{\mu}\}=\sum_{\mu\vdash n}H^{\mu}_{\lambda}e_{\mu}. \tag{3.8}$$

This is directly analogous to (3.5) and (3.6), and the coefficients appearing here may be found by a straightforward application once again of the Littlewood–Richardson rule. Clearly, in the cases for which $\boldsymbol{\lambda}=(1+a,1^b)=\binom{a}{b}$ is a single hook, $r_{\lambda}=e_{\lambda}$. This holds true for all partitions $\boldsymbol{\lambda}$ of $n=1,2$ and 3. The results appropriate to the cases $n=5$ and $n=9$ are given in table 1. For $n=5$ the complete matrix H has been displayed with rows and columns labelled by $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$ respectively, ordered with respect to the precedence convention defined in § 2. For $n=9$ the complete matrix H may be

Table 1. The coefficients H^{μ}_{λ} and J^{μ}_{λ} defined by $r_{\lambda}=\Sigma_{\mu\vdash n}H^{\mu}_{\lambda}e_{\mu}$ and $e_{\lambda}=\Sigma_{\mu\vdash n}J^{\mu}_{\lambda}r_{\mu}$ for $\lambda,\mu\vdash n=5$ and 9. An entry $x+\bar{y}$ indicates that H^{μ}_{λ} (or J^{μ}_{λ}) $=x$ whilst $H^{\tilde{\mu}}_{\lambda}$ (or $J^{\tilde{\mu}}_{\lambda}$) $=y$.

r_λ \ e_μ	5	41	31^2	21^3	1^5	32	2^21
5	1						
41		1					
31^2			1				
21^3				1			
1^5					1		
32		1	1			1	
2^21			1	1			1

e_λ \ r_μ	5	41	31^2	21^3	1^5	32	2^21
5	1						
41		1					
31^2			1				
21^3				1			
1^5					1		
32		−1	−1			1	
2^21			−1	−1			1

Table 1—continued.

$\tilde{\mu}$	1^9	21^7	31^6	41^5		2^21^5	321^4	421^3	2^31^3	32^21^2	3^21^3	42^21	2^41	32^3	3^221	
r_λ \ e_μ	9	81	71^2	61^3	51^4	72	621	521^2	63	531	52^2	431^2	54	4^21	432	3^3
9	1															
81		1														
71^2			1													
61^3				1												
51^4					1											
72		1	1			1										
621			1	1			1									
521^2				1	1			1								
63		1	1			1	1		1							
531			1	1			1	1		1						
52^2				1	1		1	1			1					
431^2				1	1			$1+\tilde{1}$				1				
54		1	1			1	1		1	1			1			
4^21			1	1			1	1		1		1		1		
432				1	1		1	$2+\tilde{1}$		1	1	$1+\tilde{1}$			1	
3^3				$1+\tilde{1}$	2		$1+\tilde{1}$	$4+\tilde{4}$		$2+\tilde{2}$	$2+\tilde{2}$	$4+\tilde{4}$		$1+\tilde{1}$	$3+\tilde{3}$	1

$\tilde{\mu}$	1^9	21^7	31^6	41^5		2^21^5	321^4	421^3	2^31^3	32^21^2	3^21^3	42^21	2^41	32^3	3^221	
e_λ \ r_μ	9	81	71^2	61^3	51^4	72	621	521^2	63	531	52^2	431^2	54	4^21	432	3^3
9	1															
81		1														
71^2			1													
61^3				1												
51^4					1											
72		−1	−1			1										
621			−1	−1			1									
521^2				−1	−1			1								
63			1	1		−1	−1		1							
531				1	1		−1	−1		1						
52^2			1	1			−1	−1			1					
431^2				$\tilde{1}$	1			$-1-\tilde{1}$				1				
54				−1	−1		1	1	−1	−1			1			
4^21				$-\tilde{1}$	−1			$1+\tilde{1}$		−1		−1		1		
432				−1	−1		1	$2+\tilde{1}$		−1	−1	$-1-\tilde{1}$			1	
3^3			$-1-\tilde{1}$		2			$-3-\tilde{3}$		$2+\tilde{2}$	$1+\tilde{1}$	$3+\tilde{3}$		$-1-\tilde{1}$	$-3-\tilde{3}$	1

recovered by noting that

$$r_{\tilde{\lambda}} = R(\tilde{\lambda}) = \sum_{\mu \vdash n} H^{\mu}_{\lambda} e_{\tilde{\mu}}, \qquad \text{i.e. } H^{\tilde{\mu}}_{\tilde{\lambda}} = H^{\mu}_{\lambda}. \tag{3.9}$$

It is to be observed that the resulting matrices H, when given completely, are lower triangular with each diagonal element equal to 1. In order to prove this it is only necessary to note two points. Firstly, the Littlewood–Richardson rule is such that the product of any irreducible character $\{\nu\}$, specified by a partition ν of Frobenius rank s, with a single-hook character $\{1+a, 1^b\} = \begin{Bmatrix} a \\ b \end{Bmatrix}$, specified by a partition of Frobenius rank 1, consists of a sum of irreducible characters specified by partitions whose Frobenius rank is either s or $s+1$. It follows that in the product

$$r_{\lambda} = R(\lambda) = R\begin{pmatrix} a_1 a_2 \ldots a_r \\ b_1 b_2 \ldots b_r \end{pmatrix} = \begin{Bmatrix} a_1 \\ b_1 \end{Bmatrix} \begin{Bmatrix} a_2 \\ b_2 \end{Bmatrix} \ldots \begin{Bmatrix} a_r \\ b_r \end{Bmatrix}, \tag{3.10}$$

which defines the expansion (3.8), the maximum value taken by the Frobenius rank of the partitions μ, specifying the terms $\{\mu\}$, is simply r.

Secondly, the Littlewood–Richardson rule implies further that in this product the terms $\{\mu\}$ which may arise are such that if

$$e_{\mu} = \{\mu\} = \begin{Bmatrix} c_1 c_2 \ldots c_s \\ d_1 d_2 \ldots d_s \end{Bmatrix}, \tag{3.11}$$

with $h_k = c_k + d_k + 1$, then the minimum value of h_1 is $a_1 + b_1 + 1$. For such terms it is necessarily true that $c_1 = a_1$ and $d_1 = b_1$, and furthermore the minimum value of h_2 is then $a_2 + b_2 + 1$, in which case $c_2 = a_2$ and $d_2 = b_2$. Continuing in this way, it is clear that the term, $\{\mu\}$, of lowest precedence, in the sense defined in § 2, is nothing other than $\{\lambda\}$ itself. This term is of maximum rank r and is lowest in lexicographic ordering with respect to the labels $\boldsymbol{h}$.

This confirms the lower triangular nature of the matrix H. Furthermore, it is clear that the matrix has a block structure, each block being associated with specific labels r and $\boldsymbol{h}$. The diagonal blocks are then necessarily seen to be unit matrices. It follows that each matrix H possesses an inverse J whose elements serve to define the inverse of (3.8), namely

$$e_{\lambda} = \{\lambda\} = \sum_{\mu \vdash n} J^{\mu}_{\lambda} R(\mu) = \sum_{\mu \vdash n} J^{\mu}_{\lambda} r_{\mu}. \tag{3.12}$$

The corresponding coefficients are also given in table 1 for $n = 5$ and $n = 9$. As in (3.9),

$$e_{\tilde{\lambda}} = \{\tilde{\lambda}\} = \sum_{\mu \vdash n} J^{\mu}_{\lambda} r_{\tilde{\mu}}, \qquad \text{i.e. } J^{\tilde{\mu}}_{\tilde{\lambda}} = J^{\mu}_{\lambda}. \tag{3.13}$$

This result may be used in the case $n = 9$ to recover from the table the complex matrix J.

4. New symmetric functions

The vector space of all homogeneous symmetric functions of degree n in the indeterminates $x_1, x_2, \ldots$ has a basis consisting of the monomial symmetric functions

$$k_{\lambda}(x) = \sum x_1^{\lambda_1} x_2^{\lambda_2} \ldots \tag{4.1}$$

for all partitions $\boldsymbol{\lambda}$ of n, where the summation is carried out over all distinct monomials in $x_1, x_2, \ldots$ with exponents $\lambda_1, \lambda_2, \ldots$ in some order (Littlewood 1940, p 63, Hall 1959, Stanley 1971).

Alternative bases are provided by the symmetric functions $h_\lambda(\boldsymbol{x})$, $a_\lambda(\boldsymbol{x})$ and $e_\lambda(\boldsymbol{x})$ encountered in § 2. These are related to $k_\lambda(\boldsymbol{x})$ by the identities

$$h_\lambda(\boldsymbol{x}) = h_{\lambda_1}(\boldsymbol{x})h_{\lambda_2}(\boldsymbol{x})\ldots \text{ with } h_m(\boldsymbol{x}) = \sum_{\mu \vdash m} k_\mu(\boldsymbol{x}), \tag{4.2}$$

$$a_\lambda(\boldsymbol{x}) = a_{\lambda_1}(\boldsymbol{x})a_{\lambda_2}(\boldsymbol{x})\ldots \text{ with } a_m(\boldsymbol{x}) = k_{1^m}(\boldsymbol{x}) \tag{4.3}$$

and

$$e_\lambda(\boldsymbol{x}) = \sum_{\mu \vdash n} K^\lambda_\mu k_\mu(\boldsymbol{x}), \tag{4.4}$$

where the coefficients constitute K^{T}, the transpose of Kostka's matrix K defined by (3.5).

Any basis which can be related to $k_\lambda(\boldsymbol{x})$ by means of a matrix with integral coefficients and determinant ± 1 is said to be an integral basis (Hall 1959, Stanley 1971). The bases provided by $h_\lambda(\boldsymbol{x})$, $a_\lambda(\boldsymbol{x})$ and $e_\lambda(\boldsymbol{x})$ are integral bases.

It follows from (3.8) and (4.4) that

$$r_\lambda(\boldsymbol{x}) = \sum_{\mu,\nu \vdash n} H^\mu_\lambda K^\mu_\nu k_\nu(\boldsymbol{x}), \tag{4.5}$$

where the matrix HK^{T} relating $r_\lambda(\boldsymbol{x})$ to $k_\lambda(\boldsymbol{x})$ possesses integral coefficients and has determinant 1, since this is true of both H and K. Thus a new integral basis is provided by the symmetric functions $r_\lambda(\boldsymbol{x})$. As an example, the functions $r_\lambda(\boldsymbol{x})$ are given in terms of $k_\lambda(\boldsymbol{x})$ in table 2 where $\boldsymbol{\lambda}$ is a partition of 5. Use has been made of the matrix H of table 1 and the known matrix K (Blaha 1969, Braunschweig and Hecht 1978), and a lexicographic ordering of rows and columns has been adopted.

There necessarily exist some further new symmetric functions, $g_\lambda(\boldsymbol{x})$ dual to $r_\lambda(\boldsymbol{x})$ defined in such a way that (Stanley 1971)

$$\prod_{i,j}(1 - x_i y_j)^{-1} = \sum_{\lambda} e_\lambda(\boldsymbol{x})e_\lambda(\boldsymbol{y}) \tag{4.6}$$

$$= \sum_{\lambda} k_\lambda(\boldsymbol{x})h_\lambda(\boldsymbol{y}) \tag{4.7}$$

$$= \sum_{\lambda} f_\lambda(\boldsymbol{x})a_\lambda(\boldsymbol{y}) \tag{4.8}$$

$$= \sum_{\lambda} g_\lambda(\boldsymbol{x})r_\lambda(\boldsymbol{y}), \tag{4.9}$$

Table 2. The coefficients $(HK^{\mathrm{T}})^\mu_\lambda$ defined by $r_\lambda = \Sigma_{\mu \vdash n} (HK^{\mathrm{T}})^\mu_\lambda k_\mu$ for $\boldsymbol{\lambda}, \boldsymbol{\mu} \vdash n = 5$.

r_λ \ k_μ	5	41	32	31^2	2^21	21^3	1^5
5	1	1	1	1	1	1	1
41		1	1	2	2	3	4
32		1	2	4	5	9	15
31^2				1	1	3	6
2^21				1	2	6	15
21^3						1	4
1^5							1

where the first equality expresses the fact that S-functions are self-dual as stressed by Hall (1959). For completeness the 'forgotten' symmetric functions, $f_\lambda(x)$ (Doubilet 1972), which are dual to $a_\lambda(x)$, have been included (Macdonald 1979, p 15).

It follows from (3.12), (4.6) and the definition (4.9) that

$$g_\lambda(x)=\sum_{\mu\vdash n} J^\lambda_\mu e_\mu(x), \tag{4.10}$$

and then from (4.4) that

$$g_\lambda(x)=\sum_{\mu,\nu\vdash n} J^\lambda_\mu K^\mu_\nu k_\nu(x). \tag{4.11}$$

In the case $n=5$ the corresponding matrix $J^{\mathrm{T}}K^{\mathrm{T}}$ relating $g_\lambda(x)$ to $k_\lambda(x)$ is given in table 3.

Table 3. The coefficients $(J^{\mathrm{T}}K^{\mathrm{T}})^\mu_\lambda$ defined by $g_\lambda=\Sigma_{\mu\vdash n}(J^{\mathrm{T}}K^{\mathrm{T}})^\mu_\lambda k_\mu$ for $\lambda,\mu\vdash 5$.

g_λ \ k_μ	5	41	32	31^2	2^21	21^3	1^5
5	1	1	1	1	1	1	1
41		1		1			−1
32			1	1	2	3	5
31^2			−1		−2	−2	−4
2^21					1	2	5
21^3					−1	−1	−1
1^5							1

Hall (1959) defined a linear transformation θ in the space of homogeneous symmetric functions through the mapping $\theta: a_m\to h_m$. This mapping is such that $\theta: e_\lambda\to e_{\tilde\lambda}$, so that θ acts as a conjugacy operation on the partitions labelling the S-functions e_λ. The action of θ on the other symmetric functions is given by

$$\theta: a_\lambda\to h_\lambda, \qquad \theta: h_\lambda\to a_\lambda, \qquad \theta: k_\lambda\to f_\lambda,$$

$$\theta: f_\lambda\to k_\lambda, \qquad \theta: r_\lambda\to r_{\tilde\lambda}, \qquad \theta: g_\lambda\to g_{\tilde\lambda}$$

where for the last two mappings use has been made of (3.9) and (3.13).

One consequence of these transformations is the validity of the identities

$$\prod_{i,j}(1+x_iy_j)=\sum_\lambda e_\lambda(x)e_{\tilde\lambda}(y) \tag{4.12}$$

$$=\sum_\lambda k_\lambda(x)a_\lambda(y) \tag{4.13}$$

$$=\sum_\lambda f_\lambda(x)h_\lambda(y) \tag{4.14}$$

$$=\sum_\lambda g_\lambda(x)r_{\tilde\lambda}(y) \tag{4.15}$$

where the first equality is a remarkable property of S-functions (Stanley 1971, Macdonald 1979, p 35).

Whilst the structure of the matrices of tables 2 and 3 makes it unlikely that the integral bases r_λ and g_λ will assume much importance, it is hoped that the expansions

(3.8) and (3.9) as exemplified in table 1 will enable calculations involving the representations of the unitary groups U(N) to be carried out using the special properties of those representations having single-hook characters.

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